Classical Solutions of Ghost Condensation Models

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Abstract

Motivated by ideas obtained from both ghost condensation and gravitational Higgs mechanism, we attempt to find classical solutions in the unitary gauge in general ghost condensation models. It is shown that depending on the form of scalar fields in an action, there are three kinds of exact solutions, which are (anti-) de Sitter space-time, polynomially expanding universes and flat Minkowski space-time. We briefly comment on gravitational Higgs mechanism in these models where we have massive gravitons of 5 degrees of freedom and 1 unitary scalar field (Nambu-Goldstone boson) after spontaneous symmetry breakdown of general coordinate reparametrization invariance. The models at hand are free from the problem associated with the non-unitary propagating mode.

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1 Introduction

It is known that general relativity successfully describes the gravitational phenomena ranging from tiny scales 10^{-4} m to huge ones 10^{12} m. On the other hand, we doubt the validity of general relativity at the Planck scale owing to appearance of space-time singularity, for which quantum effects are expected to play a dominant role. Then, from the perspective of duality of length scales it seems to be natural to ask whether general relativity might also break down at cosmological distances. Actually, general relativity cannot explain some of cosmological phenomena such as the accelerating expansion of our universe and gravitational lensing without additional inputs like still mysterious dark matter and dark energy etc. If these cosmological phenomena were purely of gravitational origin, we would have to modify general relativity in the infrared region to some extent.

One interesting approach toward such the direction is to consider the spontaneous symmetry breakdown (SSB) of general coordinate reparametrization invariance [1]-[8]. The first motivation behind this approach is that SSB might play an important role in applying string theory approach to quantum chromodynamics (QCD) in future [8]. For instance, if we wish to apply a bosonic string theory to the gluonic sector in QCD, massless fields such as spin 2 gravitons in string theory, must become massive or be removed somehow by some ingenious dynamical mechanism since such the fields do not exist in QCD. Note that this motivation is relevant to the modification of general relativity in the ultraviolet region.

The second motivation comes from the modification of general relativity in the infrared region. In addition to above-mentioned cosmological motivations, SSB of general coordinate reparametrization invariance might lead to some resolution for cosmological constant problem [6, 8]. Though the cosmological constant problem involves quantum gravity, solving the problem seems to require a low energy mechanism, for instance, a partial cancellation of vacuum energy density stemming from quantum loops. In analogy with the Higgs mechanism for conventional gauge theories, the Nambu-Goldstone mode mixes with the massless graviton, thereby changing the vacuum structure of gravitational sector in a non-trivial way.

Recently, an idea of ghost condensation has put forward in order to obtain a consistent infrared modification of general relativity [9]. In this scenario, the 'unitary' propagating scalar field appears as the Nambu-Goldstone boson for a spontaneously broken time-like diffeomorphism, and leads to the possibility for resolving cosmological problems such as inflation, dark matter and dark energy. The key point here is that a scalar ghost is converted to a normal scalar with positive-energy excitations.

More recently, 't Hooft has proposed an interesting Higgs mechanism for gravity where massless gravitons 'eat' four real scalar fields and consequently become massive [8]. In this model, all the diffeomorphisms are broken by four real scalar fields spontaneously such that vacuum expectation values (VEV's) of the scalar fields are chosen to the four space-time coordinates up to a proportional constant by gauge-fixing diffeomorphisms. Of course, the number of dynamical degrees of freedom is unchanged before and after SSB. Actually, before SSB of diffeomorphisms there are massless gravitons of two dynamical degrees of freedom and four real scalar fields whereas after SSB we have massive gravitons of five dynamical degrees

of freedom and one real scalar field so that the number of dynamical degrees of freedom is equal to six both before and after SSB as desired. Afterward, a topological term was incorporated into 't Hooft model where an 'alternative' metric tensor is naturally derived and the topological meaning of the gauge conditions were clarified [10]. ³

The problem in the 't Hooft model is that a scalar field appearing after SSB is a non-unitary propagating field so that in order to avoid violation of unitarity it must be removed from the physical Hilbert space in terms of some procedure. ⁴ A resolution for this problem was offered where one requires the energy-momentum tensor of the matter field couple to not the usual metric tensor but the modified metric one in such a way that the non-unitary scalar field does not couple to the energy-momentum tensor directly.

Combining the ghost condensation scenario with gravitational Higgs mechanism, one might be able to avoid emergence of the non-unitary scalar field in gravitational Higgs mechanism since the annoying non-unitary scalar field has its roots in one time-like component (i.e., ghost) in four scalar fields, to which the ghost condensation idea could be applied. Thus, it should be regarded that our observation is an alternative method for removing the non-unitary propagating scalar field in gravitational Higgs mechanism by 't Hooft.

In this short article, as the first step for proving this conjecture, we study classical solutions in the unitary gauge in general ghost condensation models. This analysis is needed for understanding in what background gravitational Higgs mechanism arises. It turns out that depending on the form of scalar fields in an action, there are three kinds of classical, exact solutions, which are (anti-) de Sitter space-time, polynomially expanding universes and flat Minkowski space-time.

We finally comment on gravitational Higgs mechanism in these models where we have massive gravitons of 5 degrees of freedom and 1 unitary scalar field (Nambu-Goldstone boson) after spontaneous symmetry breakdown of general coordinate reparametrization invariance. The models at hand are free from the problem associated with the non-unitary propagating mode. The detailed analysis will be reported in a separate publication.

2 Original ghost condensation model

For review and comparison with generalized models argued later, let us start with the original ghost condensation action in four space-time dimensions [9] ⁵:

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} [R - 2\Lambda + f(X)], \tag{1}$$

³Similar but different approaches have been already taken into consideration in Ref. [11].

⁴More recently, models of gravitational Higgs mechanism without the non-unitary propagating scalar field were proposed in Ref. [12].

⁵For simplicity, we have put the Newton constant G at the front, so the dimension of ghost-like scalar field ϕ^0 differs from that of the original ghost condensation model, but it is easy to modify the dimension by field redefinition.

where f(X) is a function of X which is defined as

$$X = g^{\mu\nu} \partial_{\mu} \phi^0 \partial_{\nu} \phi^0. \tag{2}$$

Recall that the ghost condensation scenario consists of three ansatzs: The first ansatz amounts to the requirement that the function f(X) has a minimum at some point X_{min} such that

$$f'(X_{min}) = 0, \ f''(X_{min}) > 0, \tag{3}$$

where $f'(X_{min})$, for instance, means the differentiation of f(X) with respect to X and then putting $X = X_{min}$. Note that the latter condition in (3) ensures ghost condensation.

As the second ansatz, the ghost-like scalar field ϕ^0 is expanded around the background mt as

$$\phi^0 = mt + \pi,\tag{4}$$

where π is the small fluctuation. This equation can be interpreted as follows: Using a time-like diffeomorphism $\delta\phi^0=\varepsilon^0$, one can take the 'unitary' gauge $\pi=0$. In other words, π is a Nambu-Goldstone boson associated with spontaneous symmetry breakdown of time translation. In this article, we consider only the case $\pi=0$ since we are interested in classical solutions.

The last ansatz is natural from the second ansatz. That is, since time coordinate t plays a distinct role from spacial coordinates x^{i} (i = 1, 2, 3), it is plausible to make an assumption on background space-time metric, which is of the Friedman-Robertson-Walker form

$$ds^2 = -dt^2 + a(t)^2 d\Omega^2, (5)$$

where a(t) is the scale factor and $d\Omega^2$ is the spatial metric for a maximally symmetric three-dimensional space.

The equations of motion are easily derived as follows:

$$\partial_{\mu}(\sqrt{-g}g^{\mu\nu}f'(X)\partial_{\nu}\phi^{0}) = 0,$$

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = T_{\mu\nu},$$
(6)

where the former equation is the ϕ^0 -equation of motion whereas the latter equation is Einstein's equations where the stress-energy tensor $T_{\mu\nu}$ is defined as

$$T_{\mu\nu} \equiv -\frac{16\pi G}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} \left[\frac{1}{16\pi G} \int d^4x \sqrt{-g} f(X) \right]$$
$$= -\left[\partial_{\mu} \phi^0 \partial_{\nu} \phi^0 f'(X) - \frac{1}{2} g_{\mu\nu} f(X) \right]. \tag{7}$$

The purpose of this paper is to look for classical solutions, so let us solve these equations of motion under the ansatzs of ghost condensation. First, note that under the second ansatz

 $\phi^0 = mt$ (recall that we put $\pi = 0$ for classical analysis), the ϕ^0 -equation of motion requires us that f(X) should be minimized at X_{min} , that is,

$$X_{min} \equiv X(\phi^0 = mt) = -m^2, \ f'(X_{min}) = 0.$$
 (8)

Then, the stress-energy tensor $T_{\mu\nu}$ becomes proportional to the metric tensor $g_{\mu\nu}$ and consequently Einstein's equations reduce to the form where the cosmological constant is shifted by this contribution

$$G_{\mu\nu} + \Lambda' g_{\mu\nu} = 0, \tag{9}$$

where we have defined the Einstein's tensor by $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ and $\Lambda' \equiv \Lambda - \frac{1}{2}f(-m^2)$. Einstein's equations at hand are nothing but the conventional form of FRW cosmology, so the solutions are respectively de Sitter, flat Minkowski, and anti-de Sitter universes according to the sign of the effective cosmological constant Λ' .

3 't Hooft model with ghost condensation

Recently, 't Hooft has proposed an interesting Higgs mechanism for gravity where massless gravitons 'eat' four real scalar fields and consequently become massive [8]. In this model, all the diffeomorphisms are broken by four real scalar fields spontaneously. A key idea is that VEV's of the scalar fields are chosen to the four space-time coordinates by gauge-fixing diffeomorphisms. Therefore, three scalar fields are usual scalar ones while one scalar field has a wrong-sign quadratic kinetic term, that is, it is a ghost field because of negative signature of the time coordinate. The problem in the 't Hooft model is that a scalar field appearing after SSB is a non-unitary propagating field, thereby leading to violation of unitarity.

In order to overcome this problem, it seems to be natural to apply the ghost condensation idea to the 't Hooft model, which is our main motivation behind the present study. Our idea, whose detail will appear in a separate publication in future, is to gauge-fix not all the diffeomorphisms but only spacial diffeomorphisms by selecting VEV's of three scalar fields to be spacial coordinates and then use the remaining time-like diffeomorphism to kill the non-unitary propagating mode. However, even in our idea, classical configuration of four real scalar fields must be taken to space-time coordinates up to an overall constant

$$\phi^a = mx^\mu \delta^a_\mu,\tag{10}$$

where the superscript a runs over 0, 1, 2, 3 as the space-time one μ . Thus, we shall take account of Eq. (10) as the classical configuration for four scalars.

A new action of ghost condensation, which is inspired by gravitational Higgs mechanism, is of form

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} [R - 2\Lambda + f(X) - g^{\mu\nu} \partial_{\mu} \phi^a \partial_{\nu} \phi^b \eta_{ab}]$$

$$= \frac{1}{16\pi G} \int d^4x \sqrt{-g} [R - 2\Lambda + F(X) - g^{\mu\nu} \partial_{\mu} \phi^i \partial_{\nu} \phi^j \delta_{ij}], \tag{11}$$

where η_{ab} is the internal flat metric with diagonal elements (-1, +1, +1, +1) and we have defined $F(X) \equiv X + f(X)$. Let us note that the last term in the first equality is added to the original ghost condensation action (1), or alternatively speaking from the side of gravitational Higgs mechanism, the third term is introduced to the original action of gravitational Higgs mechanism.

Let us seek classical solutions of this action. To do that, we first derive the equations of motion. The $\phi^i(i=1,2,3)$ -, ϕ^0 -, and Einstein's equations are respectively given by

$$\partial_{\mu}(\sqrt{-g}g^{\mu\nu}\partial_{\nu}\phi^{i}) = 0,$$

$$\partial_{\mu}(\sqrt{-g}g^{\mu\nu}F'(X)\partial_{\nu}\phi^{0}) = 0,$$

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu},$$
(12)

where the stress-energy tensor now takes the form

$$T_{\mu\nu} = (\partial_{\mu}\phi^{a}\partial_{\nu}\phi^{b} - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\partial_{\alpha}\phi^{a}\partial_{\beta}\phi^{b})\eta_{ab} - (\partial_{\mu}\phi^{0}\partial_{\nu}\phi^{0}f'(X) - \frac{1}{2}g_{\mu\nu}f(X)).$$
 (13)

Then, let us solve each equation in order. With the metric ansatz (5) and the assumption (10), the ϕ^i -equation is trivially satisfied. Next, the ϕ^0 -equation reduces to the expression

$$\partial_0(a(t)^3 F'(X)) = 0, (14)$$

whose validity requires us that one of ghost condensation ansatzs should be automatically satisfied, i.e., $F'(X_{min}) = 0$ since $X_{min} \equiv X(\phi^0 = mt) = -m^2$ and $\partial_0 X_{min} = 0$.

Finally, Einstein's equations are cast to

$$3(\frac{\dot{a}}{a})^{2} - \tilde{\Lambda} = \frac{3m^{2}}{2a^{2}},$$

$$-2\frac{\ddot{a}}{a} - (\frac{\dot{a}}{a})^{2} + \tilde{\Lambda} = -\frac{m^{2}}{2a^{2}},$$
(15)

where we have defined $\tilde{\Lambda} = \Lambda - \frac{1}{2}F(-m^2)$. Here we have made use of the expression of the stress-energy tensor which is obtained by inserting Eq. (10) to Eq. (13)

$$T_{\mu\nu} = m^2(\eta_{\mu\nu} - \frac{1}{2}g_{\mu\nu}g^{ab}\eta_{ab}) + m^2\delta^0_{\mu}\delta^0_{\nu} + \frac{1}{2}g_{\mu\nu}f(-m^2).$$
 (16)

From the equations (15), we can eliminate the terms involving \dot{a} whose result is written as

$$3\frac{\ddot{a}}{a} = \tilde{\Lambda}.\tag{17}$$

Depending on the value of $\tilde{\Lambda}$, the general solution for this equation is classified into two types. When $\tilde{\Lambda} \neq 0$, we have one type of the general solution

$$a(t) = c_1 e^{\sqrt{\frac{\tilde{\Lambda}}{3}}t} + c_2 e^{-\sqrt{\frac{\tilde{\Lambda}}{3}}t}, \tag{18}$$

where c_1 and c_2 are integration constants. However, since this solution does not satisfy the first equation in Eq. (15), it is not in fact a solution.

The other type of the general solution arises when $\tilde{\Lambda} = 0$. Then, Eq. (17) reads $\ddot{a} = 0$, so we have the general solution

$$a(t) = c(t - t_0), \tag{19}$$

where c and t_0 are integration constants. Substituting this solution into the first equation in Eq. (15), c is fixed to $c = \pm \frac{m}{\sqrt{2}}$. As a result, the line element takes the form

$$ds^{2} = -dt^{2} + \frac{m^{2}}{2}(t - t_{0})^{2}d\Omega^{2},$$
(20)

which describes the linearly expanding universe with zero acceleration. Notice that this solution is a unique classical solution, so a flat Minkowski space-time, for instance, is not a solution of this model. Here it is worthwhile to mention that although the present universe seems to be accelerating so that the linearly expanding universe is excluded from WMAP experiment, the general solution (20) might be useful in describing the future or past status of universe.

4 A general ghost condensation model

In this section, we wish to generalize the model treated in the previous section in the sense that the part including ϕ^i in the action (11) is not a linear function but a general function of $Y \equiv g^{\mu\nu}\partial_{\mu}\phi^i\partial_{\nu}\phi^j\delta_{ij}$. Through this generalization, it turns out that a flat Minkowski spacetime also becomes a classical, exact solution of the equations of motion, which would be useful in applying this generalized model to QCD in future.

The action with which we start is given by

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} [R - 2\Lambda + F(X) - H(Y)], \tag{21}$$

where 'potential' term of Y is now generalized to an arbitrary function H(Y). From this action, it is straightforward to derive the ϕ^i -, ϕ^0 -, and Einstein's equations as before, which read

$$\partial_{\mu}(\sqrt{-g}g^{\mu\nu}H'(Y)\partial_{\nu}\phi^{i}) = 0,$$

$$\partial_{\mu}(\sqrt{-g}g^{\mu\nu}F'(X)\partial_{\nu}\phi^{0}) = 0,$$

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu},$$
(22)

where the stress-energy tensor is of the form

$$T_{\mu\nu} = -\partial_{\mu}\phi^{0}\partial_{\nu}\phi^{0}F'(X) + \partial_{\mu}\phi^{i}\partial_{\nu}\phi^{j}\delta_{ij}H'(Y)$$

$$+ \frac{1}{2}g_{\mu\nu}(F(X) - H(Y)).$$
(23)

The line of argument for solving these equations of motion proceeds as in the previous section. We find that the ϕ^i -equation is trivially satisfied. The ϕ^0 -equation again leads to the condition $F'(X_{min}) = 0$, which is equivalent to one of ansatzs for ghost condensation. Then, Einstein's equations read

$$3(\frac{\dot{a}}{a})^{2} - \tilde{\Lambda} = \frac{1}{2}H,$$

$$-2\frac{\ddot{a}}{a} - (\frac{\dot{a}}{a})^{2} + \tilde{\Lambda} = \frac{m^{2}}{a^{2}}H' - \frac{1}{2}H.$$
(24)

There are some exact solutions for these Einstein's equations by choosing an appropriate form of H(Y). We shall list up only two types of exact solutions, which are polynomially expanding universes that are analogs of the linearly expanding universe (19), and a flat Minkowski space-time.

Solutions describing the polynomially expanding universes are obtained by choosing H(Y) to be

$$H(Y) = KY^n - 2\tilde{\Lambda},\tag{25}$$

where K is a constant. Then it is easy to show that the following polynomially expanding universes are exact solutions for Einstein's equations (24):

$$a(t) = \sqrt{3}m(\pm n\sqrt{\frac{K}{6}}(t - t_0))^{\frac{1}{n}},$$
(26)

which correspond to generalized solutions of the previous solution (19), for which n = 1 and K = 1.

It is remarkable that in this generalized ghost condensation model (21), a flat Minkowski space-time becomes a classical solution as well whereas it is *not* so in the 't Hooft model with ghost condensation (11). Thus, this general model opens a new avenue to gravitational Higgs mechanism, which will be utilized in QCD. In fact, with the assumptions

$$\dot{a}(t) = \ddot{a}(t) = 0, \tag{27}$$

Einstein's equations (24) reduce to the form

$$-\tilde{\Lambda} = \frac{1}{2}H,$$

$$\tilde{\Lambda} = \frac{m^2}{a^2}H' - \frac{1}{2}H.$$
(28)

Thus, as a classical solution, we have a flat Minkowski space-time if

$$H'(y) = 0, (29)$$

where $y \equiv Y(\phi^i = m\delta^i_\mu x^\mu) = \frac{3m^2}{a^2}$. This equation holds true if we select, for instance, H(Y) such that

$$H(Y) = (Y - y)^N - 2\tilde{\Lambda},\tag{30}$$

where N must be more than unity.

5 Discussion

In this article, we have found classical solutions in the unitary gauge $\pi=0$ in general ghost condensation models, which are inspired by gravitational Higgs mechanism by 't Hooft. It is shown that in the 't Hooft model with ghost condensation, there is a unique classical solution with the form of the linearly expanding universe. On the other hand, in a general ghost condensation model, we have a flat Minkowski space-time in addition to the polynomially expanding universe's solutions. Thus, this general model could lead to a candidate for gravitational Higgs mechanism.

In order to show that the general model is really physically consistent, one has to prove that the non-unitary propagating scalar mode, which appears in the original 't Hooft model for gravitational Higgs mechanism and is a serious problem, does not appear in the model at hand. Actually, we have checked that there does not appear the non-unitary scalar in our general model whose detail will be reported in a separate publication.

Although the detailed calculation is a bit involved, we can see why we have a unitary theory by a simple argument. To do so, let us count the number of the dynamical degrees of freedom before and after SSB of diffeomorphisms. Before SSB, there are massless gravitons of two dynamical degrees of freedom and four real scalars whereas after SSB we have massive gravitons of five dynamical degrees of freedom and one scalar π , so the number of the dynamical degrees of freedom is the same before and after SSB of diffeomorphisms. At this point, note that there is no room for the non-unitary propagating mode to exist in our general model. This is a reason why there is no non-unitary scalar field in this case.

However, it is still of importance to ask ourselves what symmetry is effective for killing the non-unitary mode in our model. It is a time-like diffeomorphism that kills the non-unitary mode. In this context, note that in the background ansatz (4) the time-like diffeomorphism is not broken by transforming π in a suitable manner. Thus, our general model gives us not only a new model for ghost condensation but also a nice model for gravitational Higgs mechanism.

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